September 10: Week 2 Problems

Problem 1 (Putnam 2012)

Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the interval $(1, 12)$. Show that there exist distinct indices i, j, k such that $d_i.d_j, d_k$ are the side lengths of an acute triangle.

Problem 2 (Putnam 2012)

Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

- The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;
- If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;
- If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Problem 3

a) If every point in the plane is colored either red, green, or blue, show that either some two points that are a distance of 1 apart have the same color or there is an equilateral triangle of side-length $\sqrt{3}$ that has all vertices of the same color.

b) If every point in the plane is colored either red, green, or blue, show that some two points that are a distance of 1 apart have the same color.

c) If every point in the plane is colored either red, yellow, green, or blue, show that some two points are a distance of either 1 or $\sqrt{3}$ apart and have the same color.

(You can try (c) directly, parts (a) and (b) are warm-ups to give you some hints)

Problem 4 (Putnam 2014)

A base 10 over-expansion of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, ..., 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?